Applied mathematics III

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Chapter 4
Vector Calculus
4.1 Scalar and Vector Fields

- Vector calculus concerns two kinds of functions, namely, vector functions, whose values are vectors
  $V = V(p) = (v_1(p), v_2(p), v_3(p))$
  and scalar functions, whose values are scalars
  $f = f(p)$, where p- is a point in 3-dimension.
- A vector function defines a vector field and scalar field defines a scalar field

Example:

1. Velocity, gravitational, electric and gradient are vector fields
2. $f(p) = f(x, y, z) = \sqrt{(x - x_0)^2 + (y - y_0)^2 + (z - z_0)^2}$ is scalar field

4.2 Vector Calculus

- Convergence, continuity and differentiability are the basic concept of calculus.

4.3 Curves, Arc Length and Tangent

Curves

- A curve C is a path of moving body.

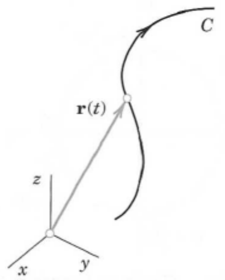

Fig. 198. Parametric representation of a curve

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- It has parameteric representation with parameter t, $r(t) = (x(t), y(t), z(t))$

Example: Give the parameteric representation of curve C along

1. Circle $(x-a)^2 + (y-b)^2 = 4, z = 0$
2. Ellipse $\dfrac{x^2}{a^2} + \dfrac{y^2}{b^2} = 1, z = 0$
3. Straight line through a point (a_1, a_2, a_3) in the direction of constant vector (b_1, b_2, b_3)
4. Straight line $y = 2x + 3, z = 7x$

Solution:

1. $C : r(t) = (a + 2cost, b + 2sint, 0)$ in CCW, positive sence or $C : r^*(t) = (a + 2cost, b - 2sint, 0)$ in CW, negative sence
2. $C : r(t) = (acost, bsint, 0)$
3. $C : r(t) = (a_1, a_2, a_3) + t(b_1, b_2, b_3) = (a_1 + tb_1, a_2 + tb_2, a_3 + tb_3)$
4. $C : r(t) = (t, 2t + 3, 7t)$

Arc

- An arc of a curve is the portion between any two points of the curve

Tangent to a Curve

- Tangents are straight lines touching a curve.
- If $r'(t) = (x'(t), y'(t), z'(t)) \neq 0$, we call $r'(t)$ a tangent vector of curve C at point P

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Fig. 193. Field of tangent vectors of a curve

- The corresponding unit vector is the unit tangent vector $u = \dfrac{r'(t)}{\|r'(t)\|}$

Example: Find the tangent to the circle $x^2 + y^2 = 4$ at $P : (\sqrt{2}, \sqrt{2})$

Solution:

$C : r(t) = (2cost, 2sint)$, $r'(t) = (-2sint, 2cost)$

The tangent to curve C is $q(t) = (\sqrt{2}, \sqrt{2}) + t(-\sqrt{2}, \sqrt{2})$

Length of a Curve

- The length of a curve $C : r(t),\ a \leq t \leq b$ is given by $l = \int_a^b \sqrt{r'(t).r'(t)}dt$

Fig. 207. Length of a curve

Arc Length of a Curve

- The arc length of a curve $C : r(\bar{t}),\ a \leq \bar{t} \leq t$ is given by $s(t) = \int_a^t \sqrt{r'(\bar{t}).r'(\bar{t})}d\bar{t}$

Example: Find th length of circular helix $r(t) = (2cost, 2sint, t)$ from $(2, 0, 0)$ to $(2, 0, 24\pi)$
Solution: $r'(t) = (-2sint, 2cost, 1)$,
$r'.r' = (-2sint, 2cost, 1).(-2sint, 2cost, 1) = 9$
Length $l = \int_0^{24\pi} \sqrt{r'(t).r'(t)}dt = \int_0^{24\pi} \sqrt{9}dt = 72\pi$
4.4 Gradient of a Scalar Field, Divergence and Curl of a Vector Field
Gradient of a Scalar Field

- The gradient of a given scalar function is the vector function defined by
  $\nabla f = (f_x, f_y, f_z)$

Example: Find the gradient of $f(x, y, z) = xy + y^2z^3$
Solution: $\nabla f = (f_x, f_y, f_z) = (y, x + 2yz^3, 3y^2z^2)$

- A vector field F is called conservative when there exists a differentiable function f such that
  $F = \nabla f$
- The function f is called the potential function for field F

Example: The vector field given by $F(x, y) = 2xi + yj$ is conservative since there exist a potential function $f(x, y) = x^2 + \frac{y^2}{2}$ such that
$F = \nabla f = 2xi + yj$
Test for conservative vector field in planes.

- Let M(x, y) and N(x, y) have continuous first partial derivatives on an open disk R
- The vector field $F(x, y) = Mi + Nj$ is conservative iff
  $M_y = N_x$

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Example: Determine whether the following vector field is conservative and if so find its potential function

1. $F(x, y) = xi + yj$
2. $F(x, y) = x^2yi + xyj$
3. $F(x, y) = 2xyi + (x^2 - y)j$

Solution:

1. $M_y = N_x = 0,\ f = x^2/2 + y^2$
   Hence, F is conservative
2. $M_y = x^2 \neq N_x = y$
3. $M_y = N_x = 2x$, this imply F is conservative
   There is f such that $\nabla f = F$
   $f_x i + f_y i = 2xyi + (x^2 - y)j$
   $\Leftrightarrow f = \int f_x dx = x^2y + g(y)$
   $\Leftrightarrow f_y = x^2 + g'(y) = x^2 - y$
   $\Leftrightarrow g(y) = -y^2/2$
   Therefore, potential function $f = x^2y - y^2/2$

Divergence of a Vector Field

- The divergence of a vector field of $F(x, y) = Mi + Nj$ in xy-plane is given by div $F(x, y) = \nabla.F = M_x + N_y$
- The divergence of a vector field of $F(x, y, z) = Mi + Nj + Pk$ in xyz-space is given by div$F(x, y, z) = \nabla.F = M_x + N_y + P_z$

Example: Find the divergence of $F(x, y, z) = x^3y^2zi + x^2zj + x^2yk$ at (2, 1, -1)

Solution: div$F(x, y, z) = \nabla.F = M_x + N_y + P_z = 3x^2y^2z$

div$F(2, 1, -1) = -12$

Curl of a Vector Field

- Curl of a vector field $F(x, y, z) = Mi + Nj + Pk$ is given by $\text{curl}F(x, y, z) = \nabla \times F = (P_y - N_z)i - (P_x - M_z)j + (N_x - M_y)k$

Example: Find the curl of $F(x, y, z) = yzi + 3zxj + zk$ with right handed

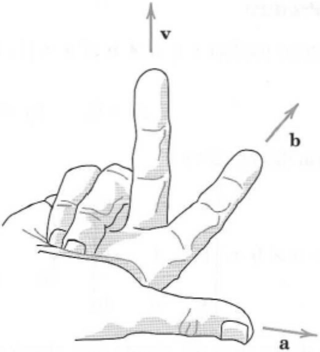


Fig. 184. Right-handed

Solution:
curl$F(x,y,z) = (P_y - N_z)i - (P_x - M_z)j + (N_x - M_y)k = -3xi + yj + 2zk$
Test for conservative vector field in space:

- The vector field $F(x,y,z) = Mi + Nj + Pk$ is conservative iff curlF=0, i.e., $P_y = N_z, P_x = M_z, N_x = M_y$

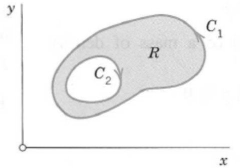
Example: If $F(x,y,z) = 2xyi + (x^2 + y^2)j + 2zk$ is conservative, find $f(x,y,z)$
Solution: curl$F(x,y,z) = (P_y - N_z)i - (P_x - M_z)j + (N_x - M_y)k = 0$
$f = \int M dx + g(y,z) = x^2 y + g(y,z)$
Hence, $f = x^2 y + yz^2$
4.5 Line Integrals, Line Integral path independent and Green's Theorem

Line Integrals

The curve C is the path of integration, $A : r(a)$ its initial point and $B : r(b)$ its terminal point

i. Let f be continuous scalar field in a region containing a smooth curve.

- If a curve C is given by $r(t) = x(t)i + y(t)j + y(t)k$, where $a \leq t \leq b$, then the line integral of f over C is given by $\int_C f(x,y,z)ds = \int_a^b f(x(t), y(t), z(t)) \|r'(t)\| \, dt$
- NB. If C_1 in CCW and C_2 in CW, then $\int_{C_1} f(x,y,z)ds = -\int_{C_2} f(x,y,z)ds$



Example: Evaluate $\int_C (x^2 - y + 3z)ds$ if $C : y = 2x, z = x, 0 \leq x \leq 1$
Solution: $C : r(t) = (t, 2t, t), 0 \leq t \leq 1, r'(t) = (1, 2, 1)$
$\int_C (x^2 - y + 3z)ds = \int_0^1 (t^2 - 2t + 3t)\sqrt{6}dt = 5\sqrt{6}/6$

ii. Let F be continuous vector field in a region containing a smooth curve.

- If C is given by $r(t) = x(t)i + y(t)j + y(t)k$, where $a \leq t \leq b$, then the line integral of F over C is given by
$\int_C F(x, y, z)ds = \int_a^b F(x(t), y(t), z(t)).r'(t)dt$
- This line integral is used to find work done by force F in the displacement along a curve $C : r(t)$

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Properties of line integral:

i. $\int_C kF.dr = k\int_C F.dr$

ii. $\int_C (F+G).dr = \int_C F.dr + \int_C G.dr$

iii. $\int_C F.dr = \int_{C_1} F.dr + \int_{C_2} F.dr, \quad C = C_1 + C_2$

Example: Find the work done (line integral) by the force $F = -xi/2 - yj/2 + k/4$ on a particle as it moves along the helix given by $r(t) = costi + sintj + tk$ from $(1,0,0)$ to $(-1,0,3\pi)$

Solution: $W = \int_C F(x(t), y(t), z(t)).r'(t)dt$

$$= \int_0^{3\pi} (-\frac{costi}{2} - \frac{sintj}{2} + \frac{k}{4}).(-sinti + costj + k)dt = \frac{3\pi}{4}$$

Line Integral path Independent

- If F is continuous on an open connected region, the line integral $\int_C F.dr$ is the path independent iff F is conservative and $\int_C F.dr = f(B) - f(A)$, where $A : r(a) \to B : r(b), \quad f$ is potential function

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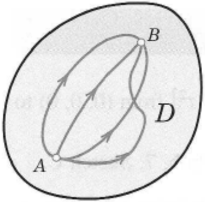


Fig. 222. Path independence

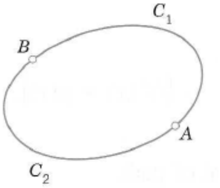
Example: For the force field given by
$F(x, y, z) = e^x \cos y i - e^x \sin y j + 2k$. Show that $\int_C F.dr$ is path independent, and find the work done by F on an object moving along a curve C from $(0, \pi/2, 1)$ to $(1, \pi, 3)$.

Solution: Since $curl F = 0$, then F is conservative.

$f = \int e^x \cos y dx + g(y, z)$

$f_y = -e^x \sin y + g'(y, z) = -e^x \sin y$

$g(y,z) = k(x,y)$
$f = e^x \cos y + k(x,y)$
$f_z = k'(x,y) = 2$
$k(x,y) = 2z$
Hence, $f(x,y,z) = e^x \cos y + 2z$
W=$\int_C F.dr = f(1,\pi,3) - f(0,\pi/2,1) = 4 - e$
Note that $\int_C F.dr = 0$ for every closed curve C in region R,
$C = C_1 \cup C_2$.



Green's Theorem (Transformation between double integral and line integral)

- Let R be a simply connected region with a piecewise smooth boundary C, oriented CCW.
- If $F(x, y) = M(x, y)i + N(x, y)j$, then $\int_C F.dr = \int_C (Mdx + Ndy) = \int\int_R (N_x - M_y)dA$

Example:

1. Use Green's theorem to evaluate the line integral $\int_C y^3dx + (x^3 + 3xy^2)dy$, where C is the path from (0,0) to (1,1) along $y = x^3$ and from (1,1) to (0,0) along $y = x$.

Solution:

W=$\int_C y^3dx + (x^3 + 3xy^2)dy = \int\int_R (N_x - M_y)dA =$
$\int_0^1 \int_{x^3}^{x} (3x^2 + 3y^2 - 3y^2)dydx = 1/4$

2. Find the work done by $F(x, y) = y^3 i + (x^3 + 3xy^2)j$ along a particle travels once around the circle of radius 3.

Solution:

$\int_C y^3dx + (x^3 + 3xy^2)dy = \int\int_R (N_x - M_y)dA = \int_0^{2\pi} \int_0^3 3r^2 \cos^2\theta r dr d\theta$
$= \int_0^{2\pi} \int_0^3 3r^2 \frac{1 + \cos 2\theta}{2} r dr d\theta = 243\pi/4$

4.6 Surface integral, Gauss Divergence theorem and its application

Surface Integrals

1. Let S be a surface with equation $z = f(x, y)$ and let R be its projection onto the xy-plane. Then the surface integral of f over S is given by

$$\int\int_S f(x, y, z)dS = \int\int_R f(x, y, g(x, y))\sqrt{1 + g_x^2 + g_y^2}dA \qquad (1)$$

2. Let S be a surface with equation $y = f(x, z)$ and let R be its projection onto the xz-plane. Then the surface integral of f over S is given by

$$\int\int_S f(x, y, z)dS = \int\int_R f(x, g(x, z), z)\sqrt{1 + g_x^2 + g_z^2}dA \qquad (2)$$

3. Let S be a surface with equation $x = f(y, z)$ and let R be its projection onto the yz-plane. Then the surface integral of f over S is given by

$$\int\int_S f(x,y,z)dS = \int\int_R f(g(y,z),y,z)\sqrt{1+g_y^2+g_z^2}dA \qquad (3)$$

Example: Find surface integral $\int\int_S yzdS$, $S:\ z = 3 - x - y$ in first-octant

Solution:

$\int\int_S yzdS = \int\int_S y(3-x-y)\sqrt{1+z_x^2+z_y^2}dA =$

$\int_0^3\int_0^{3-x}\sqrt{3}y(3-x-y)dydx = 0$

Representation of Surface

- Representation of a surface S in xyz-space is $z = f(x, y)$ or $g(x, y, z) = 0$

Example: $z = \sqrt{a^2 - x^2 - y^2}$ or $x^2 + y^2 + z^2 - a^2 = 0, \ \ (z \geq 0)$ represents a hemisphere of radius a and center 0.

- The surfaces are two dimensional.
- Hence we need two parameters u and v.
- The parametric representation of a surface S in space is of the form

$$r(u, v) = x(u, v)i + y(u, v)j + z(u, v)k,$$

where (u, v) in region R of the uv-plane

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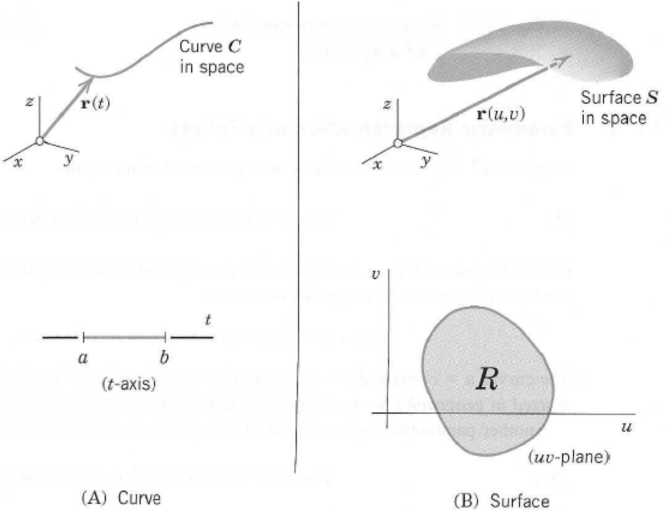

Fig. 239. Parametric representations of a curve and a surface

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Example:

1. Give the parametric representation of circular cylinder $x^2 + y^2 = a^2, \ -1 \leq z \leq 1$, has a radius a, height z and the z-axis as axis.

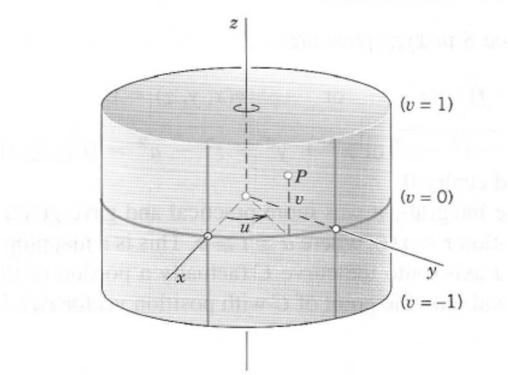

Fig. 240. Parametric representation of a cylinder

Solution:

$r(u,v) = (acosu, asinu, v),\ 0 \leq u \leq 2\pi,\ \ -1 \leq v \leq 1$ in the plane

2. Give the parametric representation of the plane $x + y + z = 1$
   Solution:
   $S:\ r(u,v) = (u, v, 1-u-v),\ 0 \leq u \leq 1,\ 0 \leq v \leq 1-u$
3. Give the parametric representation of sphere $x^2 + y^2 + z^2 = a^2$

- Note that the cross product of r_u and r_v gives a normal vector of S at P, $N = r_u \times r_v$
- The corresponding unit normal vector n of S at P is
  $n = \dfrac{N}{\|N\|} = \dfrac{r_u \times r_v}{\|r_u \times r_v\|}$,
  where r_u and r_v are tangential to S at P.

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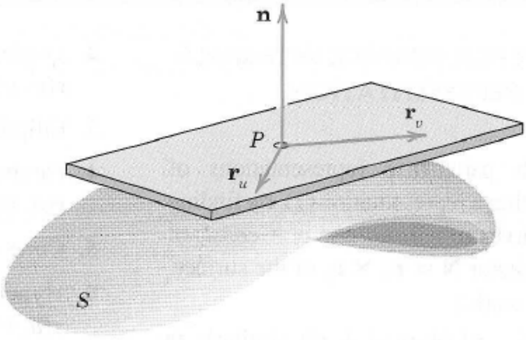

Fig. 242. Tangent plane and normal vector

Example: Find the unit normal vector to the sphere $x^2 + y^2 + z^2 = a^2$
Solution: Let $g(x, y, z) = x^2 + y^2 + z^2 - a^2 = 0$

- $N = \nabla g = 2xi + 2yj + 2zk$ is normal vector
- $n = \dfrac{N}{\|N\|} = \dfrac{xi + yj + zk}{a}$
- For a given vector function F, the surface integral of F over S by $\int\int_S F.ndA = \int\int_R F(r(u, v)).N(u, v)dudv$,
  where $N(u, v) = r_u \times r_v,\ N = \|N\|\, n,\ ndA = Ndudv$
- This integral is also called flux integral.

Example:

1. Compute the flux of water through the parabolic cylinder
   S: $y = x^2,\ 0 \le x \le 2,\ 0 \le z \le 3$ if the velocity vector is
   $V = F = 3z^2i + 6j + 6xzk$

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Solution:

- $S: \ r(u,v) = ui + u^2j + vk, \ 0 \le u \le 2, \ 0 \le v \le 3$
- $N = r_u \times r_v = (1, 2u, 0) \times (0, 0, 1) = 2ui - j, \ F(r(u,v)) = 3v^2i + 6j + 6uv$
- $\int\int_S F.ndA = \int\int_R F(r(u,v)).N(u,v)dudv = 72$

2. Evaluate the surface integral $\int\int_S (x+z)dS$, where S is the first octant portion of the cylinder $y^2 + z^2 = 9$ between x=0 and x=4.

Solution:

- In parametric form the surface is given by $r(x,\theta) = xi + 3cos\theta j + 3sin\theta k, \ 0 \le x \le 4, \ 0 \le \theta \le \pi/2$
- $r_x = i, \ r_\theta = -3sin\theta j + 3cos\theta k$
- $\|r_x \times r_\theta\| = 3$
- $\int\int_S (x+z)dS = \int\int_D (x + 3sin\theta)3dA$
  $= \int_0^4 \int_0^{\pi/2} (3x + 9sin\theta)d\theta dx = 12\pi + 36$

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Divergence theorem (Transformation between a triple integral and surface integral)

- Let Q be a solid region bounded by a closed surface S oriented by a unit normal vector n directed outward from Q.
- Then surface integral is $\int\int_S F.ndS = \int\int\int_Q divFdv$

Example:

1. Let Q be the solid region bounded by the coordinate planes and the plane 2x+2y+z=6 and $F = xi + y^2j + 2k$. Find $\int\int_S F.ndS$, where S is the surface of Q.

Solution:

- Q is bounded by four subsurfaces; S_1:xz-plane, S_2 yz-plane, S_3 xy-plane and S_4: 2x+2y+z=6
- $DivF = M_x + N_y + P_z = 2 + 2y$
- $Q = \{(x, y, z) : 0 \leq z \leq 6 - 2x - 2y,\ 0 \leq x \leq 3 - y,\ 0 \leq y \leq 3\}$

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Solution:

- $\int\int_S F.ndS = \int\int\int_Q divFdv = \int_0^3 \int_0^{3-y} \int_0^{6-2x-2y} dzdxdy = 63/2$

2. Let Q be the solid region between the paraboloid $z = 4 - x^2 - y^2$ and the xy-plane. Find $\int\int_S F.ndS$, where $F = 2zi + xj + y^2k$.

Solution:

- $\int\int_S F.ndS = \int\int\int_Q divFdv = \int\int\int_Q 0dv = 0$

3. Let Q be the solid bounded by the cylinder $x^2 + y^2 = 4$, the plane $x + z = 6$, and the xy-plane. Find $\int\int_S F.ndS$ where S is the surface of Q and $F = (x^2 + sinz)i + (xy + cosz)j + e^y k$

Solution: $Q = \left\{(x, y, z) : \ 0 \leq z \leq 6 - x, \ \ -2 \leq x \leq 2, \ \ -\sqrt{4-x^2} \leq y \leq \sqrt{4-x^2}\right\}$
$\int\int_S F.ndS = \int\int\int_Q divFdv = \int\int\int_Q 3xdv =$
$\int_0^{2\pi} \int_0^2 \int_0^{6-rcos\theta} (3rcos\theta)rdrd\theta = -12\pi$